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# Lyapunov Exponents for Infinite Dimensional Dynamical Systems

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## Abstract

Classically it was held that solutions to deterministic partial differential equations (i.e. ones with smooth coefficients and boundary data) could become random only through one mechanism, namely by the activation of more and more of the infinite number of degrees of freedom that are available to such a system. It is only recently that researchers have come to suspect that many infinite dimensional nonlinear systems may in fact possess finite dimensional chaotic attractors. Lyapunov exponents provide a tool for probing the nature of these attractors. In this paper we examine how these exponents might be measured for infinite dimensional systems.

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# 1 Introduction

An attracting set for a dynamical system is a region in phase space which 'attracts' nearby initial conditions. Any orbit started in the neighbourhood of such a set will evolve towards it and not leave it thereafter. Attracting sets for dissipative systems have dimensions which are less than those of the phase space as a whole and as they eventually trap all initial conditions it is their character which governs the long term, asymptotic behaviour of the system. It is clearly in our interest to characterize these sets as closely as we can. Attracting fixed points and periodic orbits provide two examples of 'well behaved' (and consequently dull) attracting sets. Recently however, examples have been found of some remarkable attractors. These *strange attractors* are characterized by the fact that orbits in them, which at some time lie infinitely close together, diverge from each other at an exponential rate and become uncorrelated in a finite time. Such behaviour, termed sensitive dependence on initial conditions, must be present if the system is to be considered chaotic.

Many finite dimensional flows are now known to have this sensitive dependence on initial conditions. Perhaps the best known of these is the flow of the Lorenz [1] set of three nonlinear ordinary differential equations (for appropriate parameter values). All the initial conditions quickly settle onto the attracting set but if one evolves an infinitesimal line segment of initial conditions these quickly smear out over the entire attractor. The initial conditions for any experiment, whether it be on the computer or in the laboratory, can only be specified to a finite precision and if the attractor for the system has the exponential orbital divergence property, states which are initially so close that we cannot resolve their differences will, in a finite time, give rise to quite different behaviours.

It is important to note that while we speak of exponential divergences the motions may in fact be taking place in a bounded domain. Thus two nearby orbits will diverge but may at a later time lie once more quite close together and never get infinitely far apart. The important notion here is one of the loss of predictability and it is this notion which we seek to quantify by means of the Lyapunov spectrum of the system. The rest of this paper is given over to a discussion of the Lyapunov exponents, their relevance to this problem, and in particular, their relevance to chaos in infinite dimensional dynamical systems.

## 2 Lyapunov Exponents

Lyapunov exponents (also called characteristic exponents) essentially measure the mean exponential rate of divergence of nearby orbits of a dynamical system. Let us illustrate the basic idea by considering a discrete time dynamical system, the map

$$x_{n+1} = f(x_n). \quad (1)$$

Orbits are generated by iterating on an initial seed  $x_0$ . Consider two nearby initial seeds  $x_0$  and  $x'_0$  separated by an infinitesimal distance  $\delta$ . Assuming exponential divergence at a rate  $\lambda$  (which may in general depend on  $x_0$ ) we have after  $N$  iterations,

$$\delta e^{\lambda(x_0)N} = |f^N(x_0) - f^N(x'_0)|, \quad (2)$$

whence

$$\lambda(x_0) \approx \frac{1}{N} \log Df^N(x_0), \quad (3)$$

where  $D$  represents the derivative operator. Applying the chain rule we get that

$$\lambda(x_0) \approx \frac{1}{N} \sum_{n=0}^{N-1} \log Df(x_n). \quad (4)$$

This suggests that we define the Lyapunov exponent for the map as follows,

$$\lambda(x_0) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log Df(x_n). \quad (5)$$

In this fashion we examine the linearization of the map along the trajectory generated by the seed  $x_0$ . The Lyapunov exponent  $\lambda(x_0)$  measures the mean exponential expansion/contraction rate along this orbit. Oseledec's [2] theorem can be invoked to address the question of the existence of the limit in equation (5). The interested reader is referred to [3] and the references contained therein for further details.

Next we consider a system of  $n$  ordinary differential equations,

$$\dot{x} = f(x), \quad (6)$$

where  $x = \{x_1, \dots, x_n\}$  .

Assuming the system is dissipative any initial  $n$  dimensional volume in phase space will be contracted by the flow of (6). We will consider the evolution of spheres of initial conditions centered about some orbit which we will term the fiducial (faithful) trajectory. Exponents are measured then for a particular fiducial orbit. However there is usually an assumption made about the ergodicity of the system. Roughly speaking we assume that we would get the same values for the exponents no matter which initial condition was chosen for the fiducial trajectory. For this to be true the system must have an indecomposable attractor which is visited almost everywhere by almost all orbits. That is, it must not be possible to decompose the attractor into distinct parts which might well have different properties. Moreover, almost all orbits should fill out the attractor so that the calculations (at least if performed over a long enough time span) will not depend on the particular piece of the attractor the fiducial trajectory visits.

A sphere of initial conditions will be deformed into an ellipsoid whose volume (which is proportional to the product of its semiaxes) will be smaller than the original. This does not preclude the possibility that some of the semiaxes are in fact larger than the radius of the original sphere. Initial conditions lying along such an axis will diverge from each other. Of course as the system evolves in time the semiaxes of the ellipsoid will in general not point in some constant direction but they will rotate as well as translate. This means that the detailed structure of the orbital divergences will be very complicated but the Lyapunov exponents shall be defined to give us a method of quantifying these divergences without considering all the complications

The largest Lyapunov exponent will measure the mean exponential rate of maximal orbital divergence along a trajectory. To put this another way we think of ourselves moving along our fiducial trajectory making measurements at each point. We have at our disposal an elastic (stretchable) meter stick, one end of which is tied to the trajectory with the other end being left free to evolve under the influence of the differential equation. The free end of the stick will tend to line up with the direction of maximal expansion of the system and it is the resultant change in its size that we measure. The first Lyapunov exponent is then defined as the mean value of the logarithm of our measurements along the orbit.

The higher order exponents are defined by considering the evolution not of lines but rather of volumes of initial conditions. The first two exponents will then measure the mean value of the maximal stretching of a disc of initial conditions. The circular disc deforms

into an elliptical one whose area is proportional to the product of its two semiaxes. The first two exponents refer to the mean exponential rate of change of the lengths of these axes along the fiducial trajectory.

Consider a general  $m$ -dimensional ellipsoid with time dependent semiaxes  $\alpha_1(t), \dots, \alpha_m(t)$ . The volume of the ellipsoid is proportional to  $\omega_m(t)$  where  $\omega_m(t) = \alpha_1(t)\alpha_2(t) \dots \alpha_m(t)$ . The Lyapunov exponent associated with  $\alpha_i(t)$  is then defined to be

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{\log \alpha_i(t)}{t}, \quad (7)$$

provided the limit exists.

The rate of growth of a surface element is given by the sum of the largest two exponents,  $\lambda_1 + \lambda_2$ , while the rate of growth of a  $k$ -volume element is given by the sum,  $\lambda_1 + \dots + \lambda_k$ . We note that for a dissipative system the action of the flow is to contract the phase space as a whole and therefore the sum of all the exponents will be negative.

In the infinite dimensional case we have an equation of the form,

$$\dot{u} = N(u), \quad (8)$$

where  $N$  is some nonlinear partial differential operator and  $u$  is a function of many variables. We will assume that  $u$  belongs to some normed space with a basis and for simplicity let us assume that this basis defines the boundary data for  $u$  (i.e. each basis function individually satisfies the imposed boundary conditions). We now consider the problem we had before, namely, whether nearby initial conditions evolving under the influence of (8) diverge or contract.

Distances are measured in the appropriate norm and we emphasise that any contraction or expansion is taking place in the function space and is generally not easy to interpret in physical space. Expansion can be thought of as a modal amplification with the labelling of the amplified modes defining the direction of expansion in the function space. Whereas in the finite dimensional system we might have spoken of solutions diverging in the ' $x_3$ ' direction we might now speak for example of divergences in the ' $\sin 2x_3$ ' direction.

Consider two nearby initial distributions,  $u_0(x)$  and  $u'_0(x)$  for the unknown in equation (8) evolving into the solutions  $u(x, t)$  and  $u'(x, t)$  respectively. Define  $\delta(x, t)$  to be the difference,

$$\delta(x, t) = u(x, t) - u'(x, t). \quad (9)$$

Divergence of the solutions in some direction will manifest itself as an amplification of the corresponding modes in the basis function expansion of  $\delta(x, t)$ . As in the finite dimensional case the direction of divergence may change in a complicated manner with time. That is, different modes may get amplified at different times. The Lyapunov spectrum will again be defined so as to avoid this complication of orientation and it will merely track the norm of the divergences.

In fact the exponents are defined by considering the evolution of volumes of initial conditions just as was done in the finite dimensional case. A  $k$ -sphere of initial states in function space is realized by taking any  $k$  basis functions as  $k$  separate initial conditions (assuming the basis functions are orthonormal). We can watch such a sphere evolve and deform under the influence of (8). Even though we initially take rather simple, single mode states the nonlinearities in the governing equation will generate new modes in the solutions as time evolves. In our picture the appearance of these new modes correspond to rotations of the initial sphere while changes in amplitudes of the modes correspond to deformations of the sphere into an ellipsoid.

The volume of the ellipsoid is again proportional to the product of its semiaxes with the semiaxes now being thought of as modal amplitudes. The Lyapunov exponents are defined in precisely the same manner as was done above in (7). We will discuss how such a calculation might be carried out in practise in section 5.

### 3 Dimension

It was long thought that solutions to deterministic partial differential equations could become random only through one mechanism, namely by the activation of more and more of the infinite number of degrees of freedom that are available to such a system. Recently however, researchers have come to suspect that many infinite dimensional systems may in fact possess finite dimensional chaotic attractors. For a finite set of ordinary differential equations, theoretically at least, one can measure all the characteristic exponents for the system and if their sum is negative then regions in phase space are contracting. Then not all the possible degrees of freedom are active and we might say that the attractor is of a *lower dimension* than the phase space as a whole.

The descent from an infinite dimensional phase space to a finite dimensional attractor is considerably more difficult to make. Early work in this area was done by Foias, Manley,

Temam and Treve [4]. The important question is when will a finite approximation to the full solution of a partial differential equation give qualitatively correct results. The approximations that one has in mind here consist of finite sums of appropriate basis functions. Such a numerical methodology is of course well known and goes under the general appellation of spectral methods (see for example [6]). By qualitative agreement we mean at what level of truncation will the finite approximation and the full solution display the same behaviour with regard to such properties as stability, periodicity, quasi-periodicity etc.

The approach taken is to try and show the existence of a finite dimensional Lipschitz manifold, an *inertial manifold*, for the equation in question. The distinguishing properties of this manifold are:

1. It is invariant under the flow of the system.
2. All the solutions to the equation converge exponentially to the manifold no matter where they are initiated in phase space.

Systems which have been shown to possess an inertial manifold include the one dimensional Kuramoto-Sivashinsky equation which models turbulent fluid interfaces (for an extensive treatment of this equation see [7]). This partial differential equation is then completely equivalent to a finite system of ordinary differential equations.

We note in passing that the proof of existence of the inertial manifold is reminiscent of the proof of the existence of a center manifold in the finite dimensional context. However, where the center manifold is a local construction, tied to a particular fixed point of the ordinary differential equation, the inertial manifold is a global object for the partial differential equation. It is the local nature of the center manifold which make it useful in calculations, as, close to the fixed point, it can be *approximated* by polynomial curves (a Taylor series approach). With such a specific representation of the lower dimensional surface at hand it is a relatively simple matter to unfold the system in this neighbourhood of interest and to determine the nature of the flow there (see for example [5]). The inertial manifold on the other hand is a global object which does not provide any clues as to which are the points of interest that one should try to expand about. As yet there is no systematic procedure for making use of the inertial manifold beyond saying that if it exists the system is in some strict sense equivalent to a finite dimensional system.

Any attracting sets will be subsets of the inertial manifold and it is these sets which are all important in the asymptotic limit. Unlike the manifold these sets may have no



smoothness properties to speak of. There are generalizations of the concept of dimension that may be usefully invoked when we discuss the extent and properties of these attractors. These generalized dimensions can be thought of as quantifying the amount of information that must be provided to specify a point on the attractor to within given error bounds. Extensive discussions of dimension can be found in Mandelbrot [8].

The many definitions of dimension fall for the most part into two camps, the probabilistic and non-probabilistic definitions (for a longer discussion the reader is referred to [9]). The former take into account the frequency with which the various parts of the set are visited by a typical trajectory whereas the latter merely measure metric properties of the set and ignore the dynamical features. A simple and appealing notion of a dimension which is non-dynamical in nature is that of the capacity of a set  $X$  which is defined as

$$d_C = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}, \quad (10)$$

where  $N(\epsilon)$  is the number of balls of radii  $\leq \epsilon$  needed to cover  $X$ . Intuitively we are defining  $d_C$  so that the volume of the set is proportional to  $N\epsilon^{d_C}$ . For simple sets such as points, lines and areas it is easy to verify that  $d_C$  yields the usual results of 0, 1 and 2 respectively. However for 'strange' sets  $d_C$  will yield non-integer dimensions. For instance for the well known case of the middle thirds Cantor set in one dimension,  $d_C = 0.630\dots$ , a result that fits in with the notion that this set is something less than a line and something more than a point.

To include the dynamics and the frequency of visitation in our discussion of dimension it is useful to define a quantity called the *natural measure*. Consider a ball  $B$  in the attractor and an initial condition  $x$  in the basin of attraction. Define  $\rho(x, B)$  as the fraction of time that a trajectory started from  $x$  spends in  $B$ . If  $\rho(x, B)$  is the same number for almost all  $x$  in the basin of attraction then we denote it by  $\rho(B)$  and call  $\rho$  the natural measure of the attractor.

By using such a measure to weight the balls in (10) appropriately we can ask what the capacity of the most probable part of the attractor is. In other words we can require that the  $N(\epsilon)$  balls used in (10) be chosen so that their natural measure adds up to some fraction,  $\theta$  of the measure for the full attractor (which by definition is 1). The capacity measured this way is called the  $\theta$ -capacity and is denoted  $d_C(\theta)$ . Clearly  $d_C(1) = d_C$ , however the value of  $d_C(\theta)$  for  $\theta < 1$  may be different from its value at  $\theta = 1$ . Experimentally it has been found that for many systems  $d_C(\theta)$  is independent of  $\theta$  and that  $d_C(\theta) < d_C$  for  $\theta < 1$ .

## 4 Using Exponents to Estimate Dimension

Recently there has been work done which seeks to delineate the connection between Lyapunov exponents and the dimension of attractors. A heuristic argument based on the case of an attractor having just one positive and one negative exponent,  $\lambda_1$  and  $\lambda_2$  respectively, may shed some light on a possible connection. Begin by taking a covering of the attractor with  $N(\epsilon)$  discs, each of radius  $\epsilon$ . For small enough  $\epsilon$  the action of the flow is roughly linear over the interior of the discs for some finite time  $\tau$ . In this time the discs will have deformed into ellipses with areas proportional to  $\epsilon^2 e^{(\lambda_1 + \lambda_2)\tau}$ . A finer covering of the attractor with discs of the smaller radius  $\epsilon e^{\lambda_2 \tau}$  can also be considered and it will take roughly  $e^{(\lambda_1 - \lambda_2)\tau}$  such discs to cover one of these ellipses. If we use the *assumption* that each evolved ellipse has the same area, we arrive at the estimate

$$N(\epsilon e^{\lambda_2 \tau}) = e^{(\lambda_1 - \lambda_2)\tau} N(\epsilon). \quad (11)$$

The definition of the capacity suggests that  $N(\epsilon) \approx k(1/\epsilon)^{d_C}$  and if we substitute this relation into the last equation we can solve for  $d_C$

$$d_C = 1 - \frac{\lambda_1}{\lambda_2}. \quad (12)$$

Thus motivated, we define the *Lyapunov dimension*,  $d_L$  for this case by

$$d_L = 1 - \frac{\lambda_1}{\lambda_2}. \quad (13)$$

The generalization to the higher dimensions is given by

$$d_L = m - \frac{\sum_{i=1}^m \lambda_i}{\lambda_{m+1}}, \quad (14)$$

where  $m$  is the largest integer for which  $\sum_{i=1}^m \lambda_i$  is positive.

This dimensional quantity is due to Kaplan and Yorke [10]. While we have drawn a connection between the Lyapunov dimension and the capacity, it must be noted that the Lyapunov exponents are mean quantities and, as such, are affected by the natural measure of the attractor. It has been conjectured [9], that for a typical attractor  $d_L = d_C(\theta)$  with  $\theta < 1$ .

A useful result linking the capacity and the Lyapunov exponents has been proved by Constantin and Foias [11]. This result is given in terms of the *uniform* Lyapunov exponents,

$\mu_i$  which are defined iteratively by the equation

$$\mu_i = \lim_{t \rightarrow \infty} \left( \log \frac{\omega_i(t)}{\omega_{i-1}(t)} \right) / t. \quad (15)$$

The relation puts an upper bound on the capacity as follows

$$d_C \leq \max_{1 \leq l \leq m} \left( l + \frac{\sum_{i=1}^l \mu_i}{|\lambda_{m+1}|} \right), \quad (16)$$

where  $m$  is such that  $\sum_{i=1}^{m+1} \mu_i < 0$  and strictly speaking we must take the supremum of each quantity over all the trajectories in the attractor for equation (16) to hold.

This result can be used in conjunction with the knowledge of a large enough portion of the Lyapunov spectrum to find bounds on the dimension of the attractor. The best bound will be obtained by taking  $m$  to be the last integer for which  $\sum_{i=1}^m \mu_i \geq 0$ . Thus we should like to be able to calculate Lyapunov exponents for higher and higher dimensional volumes until we find a volume that is contracted by the flow of the system and then we can use (16) to put a bound on the capacity of the attractor.

## 5 Measuring the Exponents

It is possible in theory to measure the first  $m$  Lyapunov exponents by measuring the rate of separation of  $m$  nearby trajectories. However there are many numerical problems associated with this method. The exponents supposedly probe the local structure of the attractor, being defined in terms of the mean values of some small growths in distances. Many chaotic attractors have embedded in them strange folds and twists, for example the 'double ear' structure of the Lorenz attractor. In the Lorenz case, while monitoring nearby orbits, we may at some time see a separation where one of the trajectories begins to traverse the right ear of the attractor while the others continue to travel around the left ear. At the time of separation we will witness a sudden jump in inter-trajectory distances. This jump is indicative of the global geometric properties of the attractor and it is not what we wish to quantify with the Lyapunov exponents.

A method which was discovered independently by Bennetin *et al.* [12] and Shimada and Nagashima [13] gets around such difficulties. This method advocates following just one trajectory in phase space (termed the fiducial trajectory earlier) and simultaneously

following the trajectories of  $m$  independent vectors in the tangent space to the fiducial trajectory. These evolve under the influence of the associated linearized equations. In this fashion we are looking at the evolution of an *infinitesimal* sphere of initial condition centered at all times on the fiducial trajectory. If the fiducial trajectory takes a particular path around some part of the attractor it will drag the other  $m$  linearized trajectories along that path with it. In this way we can be sure that we continue to measure local properties of the attractor at all times.

If the equation for the fiducial trajectory,  $u(t)$  is given by (8) then the equation for a *slave*, tangent space, orbit,  $v(t)$  is

$$\dot{v} = DN(u)v, \quad (17)$$

where  $DN(u)$  represents the linearization of the nonlinear operator,  $N$ , about  $u(t)$ . To calculate  $m$  exponents we solve (8) for  $u(t)$  and simultaneously solve (17) for  $m$  different initial conditions,  $v_{10}, \dots, v_{m0}$ . Tracking the divergences in  $v_1(t), \dots, v_m(t)$  gives us the exponents.

However, there are still numerical difficulties. The exponential growths in certain components of each of the tangent vectors may result in a decrease in the angle subtended by the vectors. Rather quickly on any finite precision computer these vectors will appear to become aligned and consequently are no longer independent. When this happens all of the vectors monitored will give the largest Lyapunov exponent as the rate of its growth. The familiar Gram-Schmidt orthogonalization procedure can be profitably employed to avoid this duplicity.

The modified approach is to once again solve (8) for  $u(t)$  and to simultaneously solve  $m$  copies of the linearized equations, (17) for  $v_1(t), \dots, v_m(t)$ . The initial conditions,  $v_{10}, \dots, v_{m0}$  are taken to be orthonormal. Not only that but the solutions  $v_1(t), \dots, v_m(t)$  are not let to evolve as they wish but are *reorthonormalized* every so often by means of the Gram-Schmidt procedure.

Under the flow the first vector,  $v_1(t)$  tends to line up locally with the direction currently associated with the largest Lyapunov exponent. The Gram-Schmidt procedure merely shrinks this vector and leaves its orientation unaltered. The surface element defined by the first pair of vectors,  $v_1(t)$  and  $v_2(t)$  aligns itself along the plane defined by the eigen-directions associated with the largest two exponents,  $\lambda_1$  and  $\lambda_2$ . The Gram-Schmidt procedure alters both the magnitude and the orientation of  $v_2(t)$ . However, the effect of this is to merely alter the area of the surface element leaving it aligned as it was before. Similarly, the

higher dimensional volumes will not have their orientation altered by the procedure though the defining vectors will be rotated. We remind the reader that it is the volume elements that are important for the exponent calculations.

The effect of the Gram-Schmidt procedure is twofold. It prevents the tangent vectors, which are supposedly defining volumes, from degenerating and collapsing onto each other and it also renormalizes exponentially growing numbers (vector norms), thus inhibiting overflow problems on the computer. Moreover, the norms calculated as part of the procedure are precisely the quantities we wish to monitor in our exponent calculations. Wolf *et al.* [14] can be consulted for the details of how this method can be used for finite dimensional systems.

Here we are interested in infinite dimensional systems and as an example we will now outline how the method can be used to find the exponents associated with the Kuramoto-Sivashinsky equation:

$$u_t = -4u_{xxxx} - \alpha u_{xx} - \frac{\alpha}{2}(u_x)^2. \quad (18)$$

where  $\alpha$  is a real parameter. We assume that  $u$  satisfies periodic boundary conditions in  $x$

$$u(x + 2\pi, t) = u(x, 0). \quad (19)$$

A suitable, high resolution, numerical technique for solving (18) is the Fourier collocation method. We will not go into the details of such a scheme here but merely assume that some such algorithm has been implemented on the computer and yields acceptably accurate, approximations,  $\bar{u}(x, t)$ , to the true trajectories.

The associated linearized equation for a slave trajectory  $v(x, t)$  is

$$v_t = -4v_{xxxx} - \alpha v_{xx} - \alpha u_x v_x. \quad (20)$$

and again  $v(x, t)$  is  $2\pi$  periodic in  $x$ . We assume that a suitable means of solving (20) is available and that it too yields approximate solutions  $\bar{v}(x, t)$ . The Gram-Schmidt procedure must also be implemented numerically; a simple matter if the Fourier collocation method is being used to solve for the trajectories.

With these numerical tools in place we proceed to calculate  $m$  Lyapunov exponents as follows.

1. Starting from an arbitrary initial condition we evolve  $\bar{u}(x, t)$  according to equation (18) for some time,  $T$ . The value,  $T$ , must be picked (by trial and error) to be large enough that we can feel confident that  $\bar{u}(x, T)$  is close to the attractor.

2. We then start to solve (20) for the slave trajectories. Take orthonormal initial conditions for these, for example,  $v_{k_0}(x) = \sin(kx)$ ,  $k = 1, \dots, m$ . Begin by setting the  $\bar{v}_k(x, 0)$  equal to the chosen initial conditions.
3. Update the solution  $\bar{u}(x, t)$  by one further time step using equation (18).
4. Update each slave trajectory,  $\bar{v}_k(x, t)$ ,  $k = 1, \dots, m$  for one time step using the values of  $\bar{u}(x, t)$  calculated previously, in place of  $u(x, t)$  in the governing equation (20).
5. Perform the Gram-Schmidt procedure on the  $v$ 's. Keep cumulative records of the logarithms of each of the new norms. The current estimates of the Lyapunov exponents are given by these records divided by the time elapsed since we started to solve equation (20).
6. Go to step 3.

Of course there is no way of telling when the fiducial trajectory will have settled onto the attracting set. Nor indeed is there any guarantee that there is just a single attractor. What we can do, is to experiment with various initial conditions for the fiducial trajectory and also try various values for the settle time,  $T$ . If the calculated exponents prove robust to such experimentation we can hope that they reflect the structure of the existing attractor. In any case it is our intention here to just present a tool for such investigations. We leave it for another time to consider the results that can and cannot be extracted with this tool.

We would like to conclude by considering the consequences of the existence of a positive Lyapunov exponent for an infinite dimensional system. There is a famous criterion of Hadamard for the well posedness of a partial differential equation. This essentially states that in order to be physically meaningful, equations such as (8) should be stable in the sense that small changes in the initial data should produce only small changes in the solutions. However it is now felt that there are many physical systems which do not in themselves obey this criterion. If the model equation for the system has sensitive dependence on initial conditions this may well be indicative of a physically significant phenomenon.

Nor does the the existence of some positive Lyapunov exponents doom the numericist to failure. While we are constrained at all times to a finite precision calculation and consequently cannot be certain of precisely repeating a given experiment (say of repeating a given computer run on a different machine ), we can hope to calculate accurately, mean

quantities for the system either by averaging over many runs or over large times for one run. Indeed in any practical sense such quantities are bound to be of more value than a knowledge of the detailed structure of a chaotic attractor which will be extremely complex and difficult to interpret.

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